Strong Stability Preserving Runge-Kutta method in HE-VI and split-explicit short time step integration

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1 Introduction

The Japan Meteorological Agency operates regional NWP systems using a non-hydrostatic model named ASUCA (Ishida et al. 2022), which solves fully compressive equations accommodating fast modes (e.g., acoustic waves). For fast mode calculation, ASUCA involves the use of horizontally explicit vertically implicit (HE-VI) and split-explicit time-integration (Klemp et al. 2007) with terms related to fast modes and the other terms computed in small and large time steps, respectively. To solve both, ASUCA utilizes the third-order Runge-Kutta scheme proposed by Wicker and Skamarock (2002; hereafter WS02). However, as acoustic modes typically play an insignificant role in numerical weather prediction, high-order accuracy in time integration is not necessarily required in small time steps. In this regard, the Strong Stability Preserving Runge-Kutta scheme (Shu and Osher 1998; SSP-RK) generally provides high computational stability and efficiency, but such stability has not been clarified with NWP models solving fast modes. This report outlines linear stability analysis of SSP-RK in combination with HE-VI and split-explicit time-integration schemes.

2 SSP-RK formulation

SSP-RK schemes are designed for high stability at the cost of accuracy. Here, a three-stage second-order SSP-RK scheme (hereafter, SSP-RK(3,2)) with a number of stages identical to that of WS02 was used. The formulations of the WS02 and SSP-RK schemes are shown in Table 1. As both are three-stage schemes requiring only two-stage variables (i.e., f^n and $f^{(1)}$, or f^n and $f^{(2)}$), the computational cost and memory storage requirements are almost equivalent. While WS02 accuracy is third- and second-order for linear and nonlinear terms, respectively, SSP-RK(3,2) has second-order accuracy. Von Neumann stability domains for WS02 and SSP-RK(3,2) are shown Figure 1. The latter excludes imaginary portions slightly more, but provides a larger stability domain. This suggests higher computational stability than WS02.

3 Stability analysis

In HE-VI's small time steps, horizontal advection of density and potential temperature, and pressure gradient force (PGF) terms are explicitly solved. As computation in three-dimensional models is complicated, stability analysis is carried out in a simplified one-dimensional system for each term independently.

3.1 Advection term

ASUCA computes advection term using a flux limiter scheme proposed by Koren (1993) combining third- and first-order schemes.

Table 1: Formulations of WS02 and SSP-RK(3,2) for integration of the variable f. $f^n, \Delta t$, and ϕ are f at the n-th time step, time step and a function of f, respectively.

Wicker and Skamarock(2002)	SSP-RK(3,2)
$f^{(1)} = f^n + \frac{\Delta t}{3}\phi\left(f^n\right)$	$f^{(1)} = f^n + \frac{\Delta t}{2}\phi\left(f^n\right)$
$f^{(2)} = f^n + \frac{\Delta t}{2}\phi\left(f^{(1)}\right)$	$f^{(2)} = f^n + \frac{\Delta t}{2}\phi\left(f^{(1)}\right)$
$f^{n+1} = f^n + \Delta t\phi\left(f^{(2)}\right)$	$f^{n+1} = \frac{1}{3}f^n + \frac{2}{3}f^{(2)} + \frac{\Delta t}{3}\phi\left(f^{(2)}\right)$

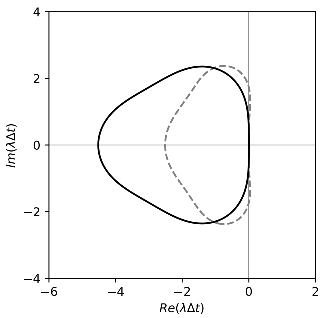

Figure 1: Stability domains for WS02(dashed grey line) and SSP-RK(3,2)(solid line).

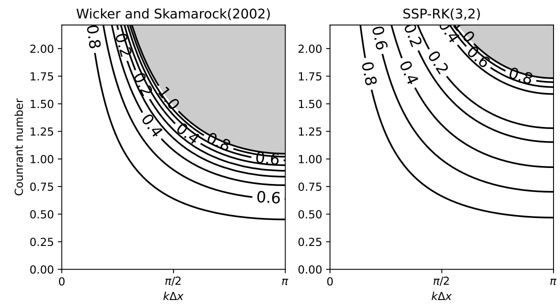

Figure 2: Amplification factors of WS02 (left) and SSP-RK(3,2) (right). The horizontal and vertical axes are $k\Delta x$ and Cr, respectively. Regions with amplification factors larger than unity (i.e., where numerical solutions are unstable) are shaded.

Appendix C of Ishida et al. (2022) shows that the critical values of the Courant number Cr of WS02 for the third- and first-order schemes are 1.61 and 1.25, respectively. In the same procedure, the critical Crs of SSP-RK(3,2) are determined as 1.25 and 2.0. As the lower values of Crs are similar, linear analysis suggests that SSP-RK(3,2) and WS02 provide comparable stability for Koren's flux limiter scheme.

3.2 Pressure gradient force term

To evaluate stabilities for PGF, linearized shallow water equations are considered as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} &= -g\frac{\partial h}{\partial x}, \\ \frac{\partial h}{\partial t} &= -H\frac{\partial u}{\partial x}. \end{aligned} \tag{1}$$

Here, these solutions are assumed:

$$\begin{aligned} u\left(x,t\right) &= \hat{u}\left(t\right)e^{ikx}, \\ h\left(x,t\right) &= \hat{h}\left(t\right)e^{ikx}, \end{aligned} \tag{2}$$

where k is a wave number. Integration of discretized forms of Eq.(1) with a time-integration scheme gives

$$\begin{pmatrix} \hat{u}^{n+1} \\ \hat{h}^{n+1} \end{pmatrix} = \mathbf{M}\begin{pmatrix} \hat{u}^{n} \\ \hat{h}^{n} \end{pmatrix}, \tag{3}$$

where $\mathbf{M}$ is an amplification matrix, which depends on the time-integration scheme, $Cr = \sqrt{gH}\Delta t/\Delta x$, and $k\Delta x$. Schemes are stable if eigenvalues (i.e., amplification factors) of $\mathbf{M}$ are less than or equal to unity for all $k\Delta x\,(\in[0,\pi])$ values. Amplification factors of WS02 and SSP-RK(3,2) can be numerically determined as shown in Figure 2. The critical Crs values for WS02 and SSP-RK(3,2) are approximately 1.05 and 1.73, respectively, indicating that the latter has higher stability for PGF. Not only in the simplified system, but also in the three dimensional model using ASUCA, we empirically confirmed SSP-RK(3,2) is stable with higher Cr than WS02.

4 Summary

Stability analysis of WS02 and SSP-RK(3,2) for the HE-VI split-explicit scheme showed that SSP-RK(3,2) provides equivalent stability for Koren's advection scheme and higher stability for PGF. As the phase velocity of the fast modes associated with PGF (i.e., acoustic waves) is much higher than advection velocity, the upper limit of the short time step is controlled by Cr for PGF. Since short time step computation is time-consuming, the higher stability of SSP-RK(3,2) for PGF enables employment of larger short time step and reduced computational cost.

References

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