Section 3

Computational studies including new techniques, parallel processing, GPUs. Effects of model resolution.

Compact finite-difference schemes for quasi-linear conservation laws

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Many physical phenomena (e.g., gas dynamics) are described by systems of quasi-linear PDE such as conservation laws. Since Richardson's famous attempt a century ago, they have been used for weather forecasting. Of course, modern models include descriptions of a huge number of additional physical processes, starting with turbulent viscosity and phase transitions. But the most important block of numerical hydrodynamic weather forecast is still an effective solver for systems of conservation laws. In the simplest case, this is the Euler – Hopf equation: $\partial_t u + \partial_x (u^2/2) = 0$.

In solutions of such equations, a so-called "gradient catastrophe" is possible — a discontinuous solution is obtained from a smooth initial condition in a finite time [1, 2].

The famous Godunov scheme [3] provides convergence of the difference scheme's solution to the exact weak solution, but this convergence is slow, and such a scheme is not very effective on smooth solutions. We need a difference scheme that provides a high order of accuracy on smooth solutions, and also converges to a weak solution after a gradient catastrophe. Typical computational artifacts are: the appearance of short–wave oscillations with a period of $2h$ ("saw-tooth" type) and/or "smoothing" of discontinuities — instead of a true strong discontinuous solution. Such TVD (total variation diminishing) schemes were described in [4]–[6] and in many others. Here h, τ — steps with respect to x and t, correspondingly.

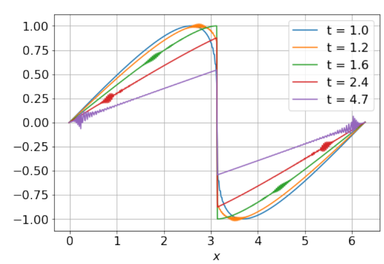

Figure 1: The solution without smoothing

Let's approximate (see [7]) the relation $\partial_t u = \partial_x f$ where $f = -(u^2/2)$, by a compact implicit difference scheme on 6-knot stencils for u and f:

$$a_1 u_{j-1}^{n+1} + b_1 u_j^{n+1} + c_1 u_{j+1}^{n+1} + a_0 u_{j-1}^{n} + b_0 u_j^{n} + c_0 u_{j+1}^{n} =$$
$$= p_1 f_{j-1}^{n+1} + q_1 f_j^{n+1} + r_1 f_{j+1}^{n+1} + p_0 f_{j-1}^{n} + q_0 f_j^{n} + r_0 f_{j+1}^{n},$$

where n and j are the indices with respect to t and x, respectively, and the coefficients that provide high accuracy order:

$$a_0 = -1,\ a_1 = 1,\ b_0 = -4,\ b_1 = 4,\ c_0 = -1,\ c_1 = 1,$$
$$p_0 = p_1 = \frac{-3\kappa}{2},\ q_0 = q_1 = 0,\ r_0 = r_1 = \frac{3\kappa}{2},\ \kappa = \frac{\tau}{h}.$$

We obtain a system of quadratic algebraic equations $\{u_j^{n+1}\}_{j=1}^{M}$, where M is the number of knots with respect to x. To avoid this very difficult computational problems we first use the explicit Euler scheme to get the first guess $\{\hat{u}_j^{n+1}\}_{j=1}^{M}$ and then introduce the representation $u_j^{n+1} = \hat{u}_j^{n+1} + \varepsilon_j^{n+1}$, where the values ε_j^{n+1} are small.

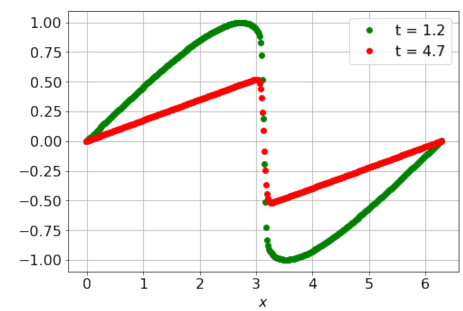

Figure 2: The smoothing of the solution, where the discontinuity is not accounted while smoothing

Let's neglect the squares of these small quantities. It turns out a tridiagonal SLAE for small perturbations $\{\varepsilon_j^{n+1}\}_{j=1}^{M}$ of order M, which is convenient to solve by the double-sweep method. If the values $\{|\varepsilon_j^{n+1}|\}_{j=1}^{M}$ are not very small, we repeat the iteration.

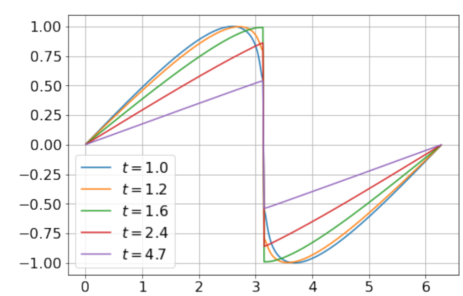

Figure 3: The solution for the full compact scheme

The zone with a large gradient of the solution $u(t, x)$ emits a wave packet of short waves of the "saw-tooth" type, Fig. 1. To dampen these artificial computational waves, we use special grid smoothing operators that are close to the identical operator on long waves and suppress "saw-tooth" type waves. Explicit or implicit (compact) grid smoothing operators are

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used. This ensures the correct behavior of the total variation ($TV(n) = \sum_j |u^n_{j+1} - u^n_j|$) and the first integrals of the finite difference solution. Fig.2–5.

It is possible to smooth the solution on the entire periodic grid to suppress the "saw-tooth" type oscillation. However, the jump itself will be smoothed too (it is smeared and its amplitude decreases), see Fig.2. It is better to smooth the solution separatly for each interval between jumps (in our example, this is one interval); see Fig.3 for the more exact solution.

We have choiced experimentally the parameters of these smoothing operators to obtain a scheme with effective damping of artificial short waves and also does not allow smoothing of the discontinuity, Fig.3. The solution with initial data $u(0, x) = \sin(x)$, $x \in [0, 2\pi)$ was used for these experiments.

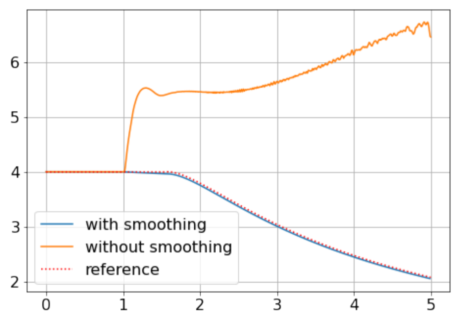

Figure 4: The total variation of the solutions of the compact difference scheme with/without smoothing operator and for the reference weak solution

The integral functionals of the form $I_g[u] = \int_0^{2\pi} g(u(t, x))\, dx$, are constant as long as the solution $u(t, x)$ of the Euler – Hopf equation with the periodic boundary conditions is smooth. Here g is an arbitrary smooth function.

After the gradient catastrophe, only the first integral for $g(u) = u$ is preserved, and the other ones decrease. For the compact scheme with suitable choice of the smoothing operators, the evolution of the first integrals is exactly the same, Fig.5.

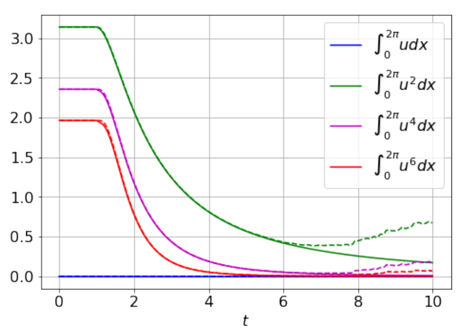

Figure 5: The dynamics of the functionals $I_k[u] = \int_0^{2\pi} u^k(t, x)\, dx$ for the compact scheme (dotted line — the compact scheme without smoothing)

Conclusion. The compact implicit finite-difference "predictor – corrector" scheme demonstrated high accuracy not only for smooth solutions, but for non-smooth and discontinuous ones, too. We apply on every temporal step first explicit Euler scheme and second effective double-sweep method to determine perturbations according to the compact high order scheme. Then the smoothing maps damp artificial short waves in the solution. The approach can be applied for systems of conservation laws as well as for such equations together with linear diffusion.

References

[1] G.B.Whitham. Linear and Nonlinear Waves. 1999 John Wiley & Sons, Inc.

[2] P.D.Lax Weak Solutions of Nonlinear Hyperbolic Equations and Their Numerical Computation. Commun. Pure and Appl. Math., 1954, V.7(1), PP.159–193

[3] S.K.Godunov Reminiscences about difference schemes. J. Comput. Phys. 1999, 153(1), PP.6-25.

[4] A.Harten, J.M.Hyman, P.D.Lax On finite difference approximations and entropy conditions for shocks. Commun. Pure Appl. Math. 1976, V.29, PP. 297–322.

[5] A.Harten High resolution schemes for hyperbolic conservation laws. J. Comput. Phys. 49(3), 1983, PP.357–393. V.135, №2, 1997, PP.260-278.

[6] H.C.Yee. Construction of Explicit and Implicit Symmetric TVD Schemes and Their Applications. J. Comput. Phys., V. 68, PP.151–179, 1987.

[7] V.A. Gordin. Compact finite-difference scheme for differential relations' approximation. Math. Models and Computer Simulations, 2020, 20(2), PP.133–142.

[8] Ph.L. Bykov, V.A.Gordin. Objective Analysis of the Structure of Three-Dimensional Atmospheric Fronts. Izvestiya Russian Academy, Atmospheric and Oceanic Physics, 48(2), 2012, PP. 152–168.