Integration of the Euler equations with the exponential method.

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Most of the time integration schemes used in meteorological models are either semi-implicit or split-explicit. These choices are predetermined by a fundamental work of A. Robert and G. Marchuk from the late 1960s. The robustness and efficiency of both algorithms is undisputed, as proven by numerous examples of Numerical Weather Prediction (NWP) models [2]. In the last decade several attempts have been made to design the alternative methods of the time stepping (see the introductions in [1] and [2] for review). One of the emerging techniques is the exponential time integration, which is derived from the general principles of the theory of dynamic systems. The essence of the exponential method lies in the dynamic linearization of the meteorological equations combined with the analytical evaluation of a resolvent operator. The example of implementation of such a method in the icosahedral shallow water model is described in [1]. The main conclusion from this study is that the time step used can be increased six-fold with respect to the semi-implicit method, while still increasing accuracy of the model. The issue of efficiency of the exponential scheme was addressed in the separate study, which shows that the execution time may be comparable to the semi-implicit scheme, provided that some optimizations of the code are introduced [2] and [3]. The important question is whether or not the same results can be achieved for the compressible Euler equations. In our study, we use the equations for perturbations with respect to the height dependent reference state. In standard notation, the system under consideration can be written as follows

$$
\begin{aligned}
\partial_t \mathbf{v} &= -\mathbf{v}\nabla\mathbf{v} - c_p(\overline{\theta}+\theta')\nabla\pi' + g(\theta'/\overline{\theta})\hat{\mathbf{z}} \\
\partial_t \pi' &= -\mathbf{v}\nabla\pi' - (R/c_v)(\overline{\pi}+\pi')\mathrm{div}\mathbf{v} + g(\mathbf{v}\cdot\hat{\mathbf{z}})/c_p\overline{\theta} \\
\partial_t \theta' &= -\mathbf{v}\nabla\theta' - (\mathbf{v}\cdot\hat{\mathbf{z}})\partial_z\overline{\theta}
\end{aligned}
$$

Fluid equations are discretized on a Cartesian grid using the fourth-order central finite difference approximation for space derivatives. Bottom no-flow through and cyclic lateral boundary conditions are imposed. The resulting set of semi-discrete Ordinary Differential Equations (ODEs) is solved with the exponential integration method with the analytic Jacobian as in [1]. The equations considered in this study are inviscid and the numerical noise is controlled with the help of the Shapiro filter.

We have selected two different initial conditions. In the first case, we consider two air bubbles with a small positive and negative buoyancy placed at different levels, while in the second case, a single bubble with negative buoyancy placed just below the upper lid. The first initial condition leads to a collision of the thermals, whereas the second one generates the gravity current splashing at the ground. The grid system used in the simulation of colliding thermals extends from the surface up to 5 km and has a horizontal length of 5 km, the corresponding sizes for the gravity current are 3 and 8 km respectively. The numerical mesh in both cases was homogeneous with a constant spacing of 20 m. All calculations in both cases were performed with the time step of 15 seconds giving the Courant number for acoustic waves around 250.

The results of the simulation of colliding thermals are depicted in Fig. 1. Evidently, the method is stable despite very long time steps and can capture a sharp contrast between the cold and warms bubbles. This observation is further reinforced by the results for the gravity current shown in Fig. 2, just at the time of impact and at a later stage when the flow changes direction from horizontal to vertical. The execution time for a single time step of the MATLAB code per grid point is about 0.00007 sec on a single 2.9 GHZ Intel I9 core. The future work with the presented model will include testing of the other approximations of the space derivatives and the experiments with moist thermodynamics.

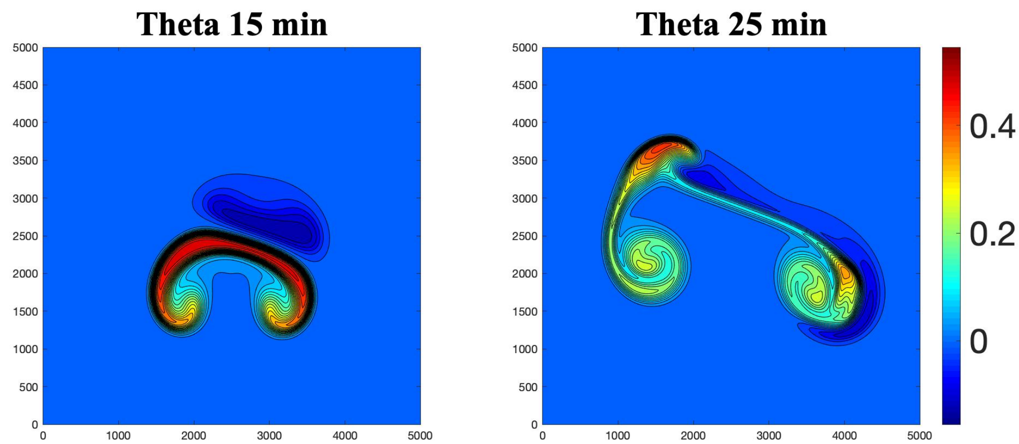

Fig. 1 Potential temperature perturbation in the convective element colliding with a negative buoyancy bubble.

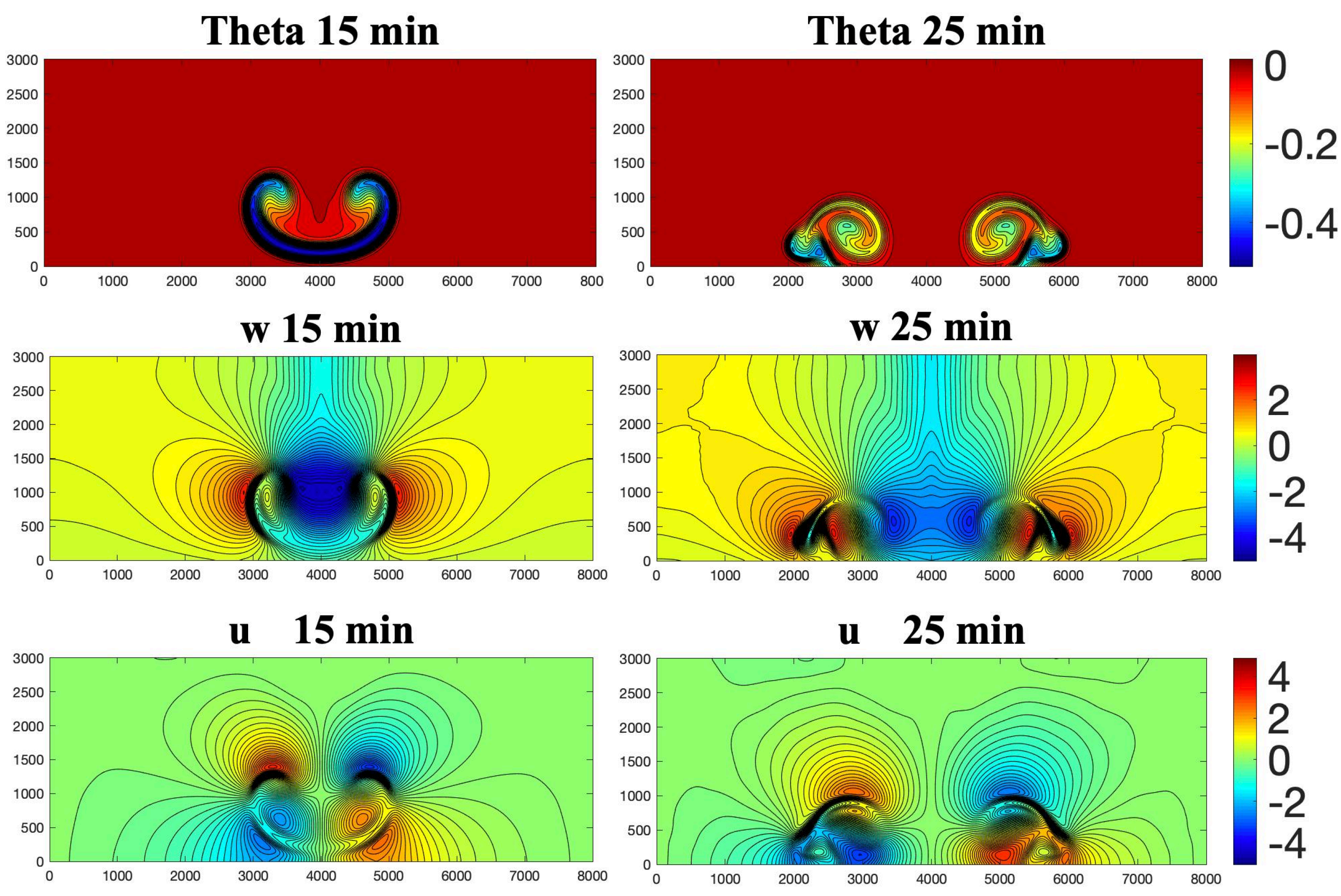

Fig. 2 Potential temperature perturbation and velocity components (m/s) in the descending cold bubble splashing on the surface.

References

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