A FORMAL APPROACH FOR SMOOTHING ON VARIABLE-RESOLUTION GRID
Part II: Filtering the scalars on the polar grid

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The filter formulation developed in 1D and generalized in 2D Cartesian domain (Surcel and Laprise 2010) is here adapted to 2D cylindrical polar geometry, as a step towards spherical polar geometry. The filter is tested for different scalar test-functions, first to control the "pole problem" specific of the latitude-longitude models, and second to remove the anisotropic noise outside the high-resolution area of a polar stretched grid.

On the polar grid the formulation of the filter is obtained by separate applications of the filter in radial and azimuthal directions. The filtered function can be written in integral form as

$$\overline{\psi}(r,\lambda) = \overline{\left(\overline{(\psi)}^{\lambda}\right)}^{r}(r,\lambda) = \int_{r'=0}^{\infty} \int_{\lambda'=0}^{2\pi} \psi(r',\lambda')\, w\left(r'(\lambda-\lambda')\right) w(r-r')\, r'\, dr'\, d\lambda' .$$

On a discrete polar grid (r_i,λ_j), the field to be filtered is represented by $\psi_{i,j} = \psi(r_i,\lambda_j)$, with $i = 1,\ldots,n; j = 1,\ldots,m$ and $r_i \in [0,R_e], \lambda_j \in [0,2\pi[$ where R_e is the distance from the centre of the grid to the boundary (henceforth referred to as the equator). Then the convolution formula is discretized as follows:

$$\overline{\psi}^{r,\lambda}(r_i,\lambda_j) = \frac{\sum_k \overline{\psi}^{\lambda}(r_k,\lambda_j)\cdot w\left(d^r_{i-k}\right)\cdot s(r_k)}{\sum_k w\left(d^r_{i-k}\right)\cdot s(r_k)} = \frac{\sum_k \sum_l \psi(r_k,\lambda_l)\cdot w\left(d^{\lambda}_{j-l}\right)\cdot w\left(d^r_{i-k}\right)\cdot s(r_k)\cdot s(\lambda_l)}{\sum_k \sum_l w\left(d^{\lambda}_{j-l}\right)\cdot w\left(d^r_{i-k}\right)\cdot s(r_k)\cdot s(\lambda_l)}$$

where $d^r_{i\text{-}k}$ and $d^{\lambda}_{j\text{-}l}$ are the radial and azimuthal distances, and $s(r_k)\cdot s(\lambda_l) = s(r_k,\lambda_l)$ is the unit area surface. It must be noted that the weighting function varies with the physical distances, and is not based on grid-point count, which is in fact the critical ingredient in the design of the proposed convolution filter.

To test the skill of the filter at alleviating the "pole problem", we employ a discrete uniform polar grid. The test functions are composed of a large-scale field referred to as the signal or physical component and a small-scale field referred to as the noise. The large-scale signal will be represented by a double cosine in physical space or by a cylindrical harmonic (eigenfunction of the Laplacian on the polar grid). The small-scale noise will be represented by a double cosine in physical space.

A first example regarding the application of the filter to remove the "pole problem" is presented in Fig. 1a. The test function is composed from a large-scale signal in form of a double cosine with wavelength $20{,}000\,\mathrm{km}$ and a similar noise with wavelength of $500\,\mathrm{km}$. The convolution filter is defined such as to keep unchanged all scales larger than $2{,}400\,\mathrm{km}$ and to remove all scales smaller than $800\,\mathrm{km}$. With these parameters the quadrature requires a minimum truncation distance of $1{,}600\,\mathrm{km}$ for an adequate accuracy of the convolution. The second example presented in Fig. 1b uses a large-scale signal in form of cylindrical harmonic with radial and azimuthal wavenumbers equal to 2 and a noise with wavelength of $600\,\mathrm{km}$. For this example the filter is designed such as to keep unchanged all scales larger than $2{,}400\,\mathrm{km}$, but to remove all scales smaller than $1{,}000\,\mathrm{km}$. Because the weighting function corresponds to a more abrupt response function, the convolution needs a larger truncation distance of $2{,}300\,\mathrm{km}$ to properly remove the noise and to preserve the large-scale signal. To quantitatively assess the influence of the cut-off distance, three different weighting functions were tested. The normalized root mean square error (*NRMS*) is computed between the filtered solution and the expected analytical solution. The parameters characterizing these weighting functions are $w1(L_a = 2{,}400\,\mathrm{km}; L_b = 1{,}000\,\mathrm{km})$, $w2(L_a = 2{,}400\,\mathrm{km}; L_b = 800\,\mathrm{km})$ and $w3(L_a = 2{,}400\,\mathrm{km}; L_b = 600\,\mathrm{km})$, where L_a represents the minimal wavelength to be preserved and L_b the maximal wavelength to be removed by the filter. The *NRMS* curve represented in Fig.1c for truncation distance between $200\,\mathrm{km}$ and $2{,}400\,\mathrm{km}$ shows that the error decreases as the truncation distance increases, although not monotonically. The oscillations are larger for $w1$ than for $w3$ due to Gibbs' phenomenon associated with the narrower response function, which necessitates wider stencil for accurate representation.

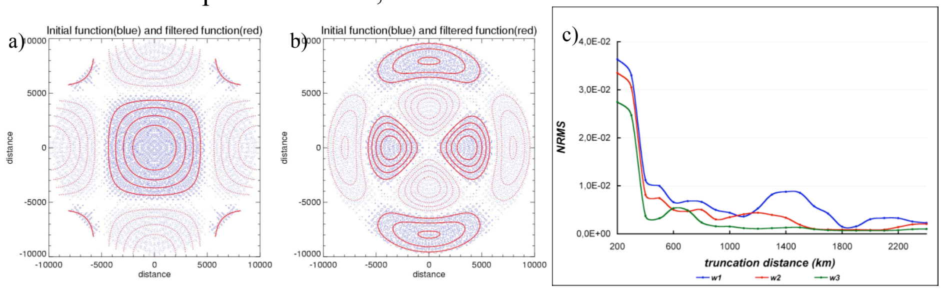

Figure 1: An initial function shown in blue and the filtered function is shown in red. The filter uses the weighting function w2 (a) and w1 (b) and truncation distances of 1,600km (a) and 2,300km (b). c) The NRMS curves as function of the truncation distance for the three weighting functions presented above.

We now present the skill of the filter at removing the anisotropy on the stretched polar grid. This grid contains a "uniform" high-resolution domain in the sector $(r_1, r_2) = (3{,}500; 7{,}500)$ km and $(\lambda_1, \lambda_2) = (5/8 \cdot 2\pi, 7/8 \cdot 2\pi)$. A gradual stretching zone is used adjacent to the high-resolution area: $r \in (2{,}500; 3{,}500) \cup (7{,}500; 8{,}500)$ and $\lambda \in (4.5/8 \cdot 2\pi, 5/8 \cdot 2\pi) \cup (7/8 \cdot 2\pi, 7.5/8 \cdot 2\pi)$, with local stretching rate of $s_r = 8\%$ in the radial direction and $s_\lambda = 3.8\%$ in the azimuthal direction. Low resolution is used elsewhere in the domain resulting in a total stretching factor of $S_r \cong S_\lambda \cong 6$.

An example of the application of the filter on the polar stretched-grid is presented in Figs 2a and 2b. The first panel represents the initial test function composed from a cylindrical harmonic with radial and azimuthal wavenumbers equal to 3 and a noise in form of a double cosine with wavelength 400km. The noise is gradually added in the stretching zones and in the high-resolution area and the filter is applied outside the uniform high-resolution region. The convolution filter uses the weighting function $w1$ and a truncation distance of 2,300km. We note that the noise is completely removed and not deformations or attenuations of the large-scale signal were observed. We verify the performance of the filter to conserve the filter quantities computing the normalized conservation ratio (*NCR*) between the initial and the filtered function. Figure 2c presents the *NCR* for a test function similar with those used for the uniform grid, but with the noise added gradually in the stretching areas and the high-resolution domain. Using the same weighting functions as in the first case we note a better conservation for weighting functions that need a smaller truncation distance, but all *NCR* curves eventually approach zero when using large truncation distances.

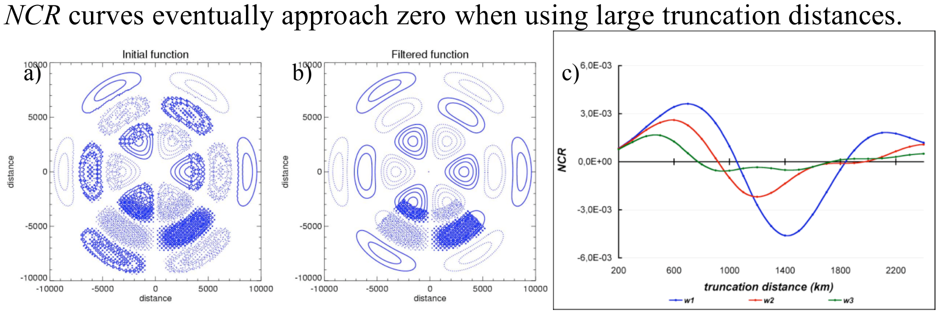

Figure 2 The initial (a) and the filtered (b) signal when the convolution filter uses the weighting function w1 and a truncation distance of 2,300km. c) The NCR curves as function of the truncation distance for the three weighting functions presented above.

The experiments realized on the polar grid demonstrated the ability of the convolution filter to adequately remove small-scale noise both in the polar region and also in the anisotropic "arms-of-the-cross" regions of the variable polar stretched grid. The convolution filter can be concomitantly applied to address the pole problem and also to remove anisotropic noise in the stretching region of the grid, by choosing appropriate parameters for the convolution weighting function.

Reference

Surcel, D., and R. Laprise, 2010: A General Filter for Stretched-Grid Models: Application in Cartesian Geometry. *Mon. Wea. Rev.* doi: 10.1175/2010MWR3531.1.