EXPERIMENTS WITH REGULARIZATION OF THE GEODESIC ICOSAHEDRAL GRID

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The geodesic grid obtained from the iterative process of division of the icosahedron imbedded in a sphere offers certain advantages in the simulation of fluid flow on the sphere such as quasi-homogeneity and elimination of the pole problem typical for the standard latitude-longitude meshes. The ratio of the maximum to the minimum mesh interval ($\gamma = \delta_{\max}/\delta_{\min}$) for the icosahedral grid is slightly higher than one (1.2 for grid of level 4). This property is not unusual in the practice of computational fluid dynamics where variable grids are used quite commonly. Unfortunately, when analyzing the distribution of the grid interval on the surface of the sphere one can notice the pronounced discontinuities of the mesh interval as depicted in Fig. 1. Sudden changes of the grid properties can lead to generation of spurious vorticity, and consequently, to systematic errors when solving the partial differential equations on geodesic grid.

In order to alleviate this problem, the grid have to be regularized. One of the most popular options is the application of the, so called, spring dynamics (Tomita et al., 2001) in which the nodal points of the grid are connected by springs creating thus a large set of coupled harmonic oscillators. The time evolution of such system is governed by the following system of equations

$$\sum_{j(i)} k((\|\mathbf{r}_i - \mathbf{r}_j\| - \hat{d})\mathbf{e}_{ij} - \alpha \mathbf{w}_i = M_i \frac{d\mathbf{w}_i}{dt}, \quad \mathbf{w}_i = \frac{d\mathbf{r}_i}{dt} \tag{1}$$

where $\mathbf{r}_i$ is the position of the i-th node, $M_i = 10^4$, $k = 10^{-4}$, $\alpha = 10$, $\mathbf{e}_{ij} = \mathbf{r}_j - \mathbf{r}_i$, $\hat{d} = (2\beta\pi R)/(10 \times 2^{l-1})$ is the equilibrium spring length, $\beta = 0.4$ and l is the number of partitions of the icosahedron used to generate grid. The oscillating nodal points equilibrate after some time depending on the selected values of the parameters. The distribution of the nodal points obtained after the system achieves a state of the minimum energy is much more regular than that for the standard grid. The sample result for the grid of level four after 250 time steps with the 4-th order Runge-Kutta scheme ($\triangle t = 100$ s.) when the masses ceased to move is depicted in Fig. 2. The regularization of the grid, however, is associated with the increase of γ (for the case of grid level four γ increases from 1.2 to 1.5). This problem can be reduced by the use of nonlinear spring rates as suggested by Tomita et al. (2001). The alternative approach to the icosahedral grid regularization is based on the fundamental theorem of differential geometry. According to the Theorema Egregia of Gauss, the Laplace-Beltrami operator on manifold $\mathcal{M}$ can be expressed as follows

$$\triangle_{\mathcal{M}}\, p = -2H(p) \in \mathbf{R}^3 \tag{2}$$

where $H(p)$ is the mean curvature normal at $p \in \mathcal{M}$, and $\mathcal{M}$ is the two dimensional manifold in $\mathbf{R}^3$. After relation (2) is discretized on a standard icosahedral grid approximating $\mathcal{M} \equiv \{\mathbf{r} \in \mathbf{R}^3 : \|\mathbf{r}\| = a\}$, there is a relatively pronounced error caused by the sudden changes of the mesh interval depicted in Fig. 1. The error of approximation of (2) can be reduced by the proper redistribution of the nodal points using the following iteration

$$p_i^{n+1} = -\left(\frac{a^2}{2}\right)\mathbf{L}\, p_i^n \tag{3}$$

where $p_i^k \in \mathbf{R}^3$, a is the radius of the sphere, and $\mathbf{L}$ the approximation of the Laplace-Beltrami operator. The result of grid regularization obtained after 20 iterations of (3) is shown in Fig. (3). The value of γ is 1.24 for the grid of level four which is slightly better than that for the case of spring dynamics even with the nonlinear relation for k. The further comparison of both methods is currently investigated in order to establish the best grid regularization technique.

The quality of a regularized grid can be verified by investigation of the approximation error of ∇, div and $\nabla\times$ operators. We use the test functions $\alpha(\lambda,\theta) = \sin(\lambda)$, $\beta(\lambda,\theta) = \cos(m\lambda)\cos^4(n\theta)$, $\mathbf{u} = \alpha\nabla\beta$ with $m = 3$ and $n = 3$ and calculate $\nabla\,\mathbf{u}$, $\hat{\mathbf{k}}(\nabla\,\times\mathbf{u})$ and $\nabla\beta$ using the approximation described in Pudykiewicz (2006). The convergence of l_2 and l_∞ error norms for these operators on grids regularized by iterating relation (3) is shown in Figs. 4–6 (l_2 is indicated by squares, l_∞ by triangles, the results for standard icosahedral grids are denoted by dashed lines; the red line indicates the second order convergence).

REFERENCES

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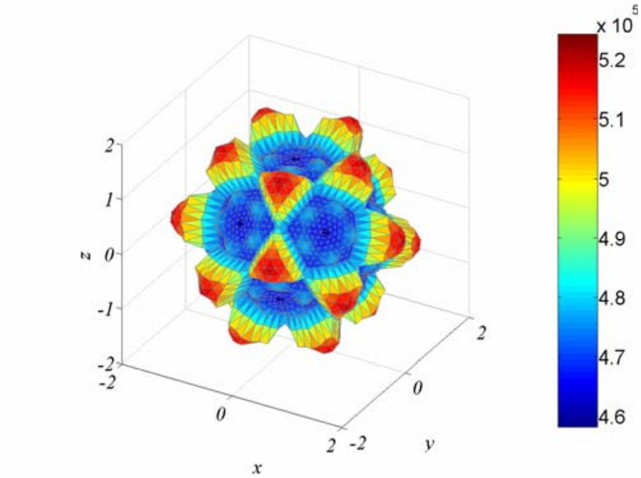

Fig.1 Surface: $(1+(\delta-\delta_{\min})/(\delta_{\max}-\delta_{\min}))$ for the grid of level 4; (δ is the mesh interval)

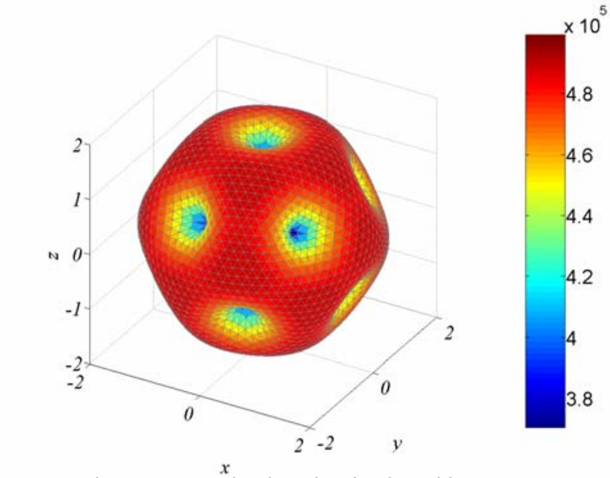

Fig. 2 Same as in Fig. 1 but for the grid regularized using the spring dynamics method

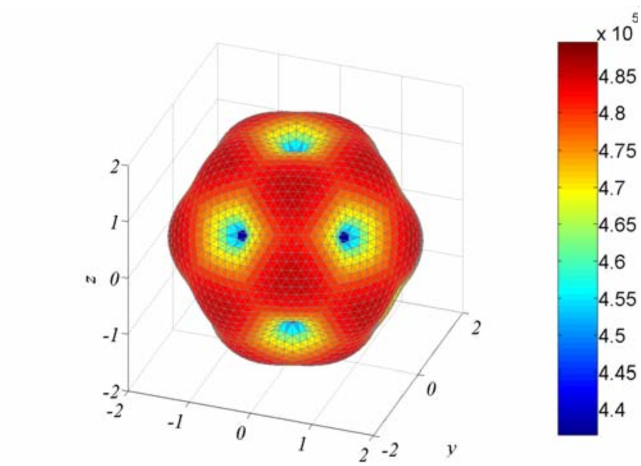

Fig. 3 Same as in Fig. 1 but for the grid regularized using the relation resulting from Gauss "Theorema Egregia"

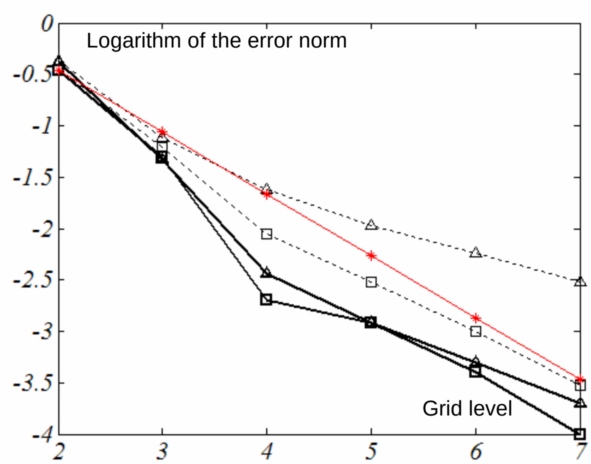

Fig. 4 The convergence of error norms for the gradient operator

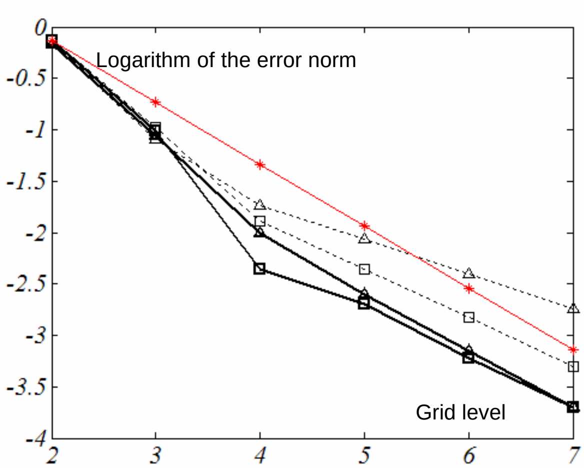

Fig. 5 The convergence of error norms for the divergence operator

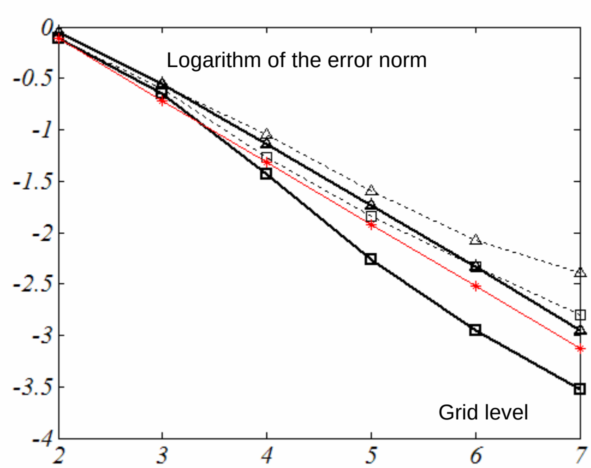

Fig. 6 The convergence of error norms for the rotation operator