FAST ALGORITHM OF HARMONIC ANALYSIS OF REAL-VALUED FUNCTIONS ON A SPHERE

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Let us consider a sphere of unit radius centered at the origin. Let us take the coordinate system using the following independent variables: co-latitude θ ($\theta = \pi/2 - \varphi$, where φ is the latitude), longitude λ, and radial coordinate ρ. We introduce the notation $x = \cos\theta$.

Let us consider the Jacobi polynomials on the interval [-1, 1]. They are orthogonal with a weight function $h(x) = (1-x)^{\alpha}(1+x)^{\beta}$ ($\alpha > -1$, $\beta > -1$) [1,2]. The integrability of function $h(x)$ is provided by the conditions $\alpha > -1$, $\beta > -1$.

At $\alpha = \beta$ the Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$ turn into the so-called ultraspherical polynomials [2], which satisfy the hypergeometric equation

$$\left(1-x^2\right)y'' - 2(\alpha+1)xy' + n(n+2\alpha+1)y = 0 . \qquad (1)$$

In this case, the weight function $h(x)$ is even and the ultraspherical polinomials $P_n^{(\alpha,\alpha)}(x)$ will be even or odd depending on eveness or oddness of its order n, that is, $P_n^{(\alpha,\alpha)}(x) = (-1)^n P_n^{(\alpha,\alpha)}(-x)$.

Let $\alpha = \nu - 1/2$ ($\nu > -1/2$). According to [2], let us introduce the following notation and normalization:

$$P_n^{(\nu)}(x) = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\frac{\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+1)}P_n^{(\alpha,\alpha)}(x) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(2\nu)}\frac{\Gamma(n+2\nu)}{\Gamma\left(n+\nu+\frac{1}{2}\right)}P_n^{\left(\nu-\frac{1}{2},\nu-\frac{1}{2}\right)}(x). \qquad (2)$$

Note the following important particular cases of ultraspherical polynomials. The equation

$$\left(1-x^2\right)T_n'' - xT_n' + n^2 T_n = 0 , \qquad (4)$$

follows from (2) at $\nu = 0$. Its solutions are the first kind Chebyshev polynomials $T_n(x) = P_n^{(0)}(x)$.

At $\nu = 1$ Eq. (2) reduces to the equation

$$\left(1-x^2\right)U_n'' - 3xU_n' + n(n+2)U_n = 0 , \qquad (5)$$

the solution of which are the second kind Chebyshev polynomials $U_n(x) = P_n^{(1)}(x)$. The third particular case at ν = 1/2 gives the solution of Eq. (2), which is called the Legandre polynomials, $L_n(x) = P_n^{(1/2)}(x)$.

Note that $P_n^{(\nu)}(\cos\theta)$ can be represented in the form of the Fourier expansion in orthogonal first-kind Chebyshev polynomials [2]:

$$P_n^{(0)}(\cos\theta) = \cos n\theta,$$
$$P_n^{(\nu)}(\cos\theta) = 2\gamma_0\gamma_n \cos n\theta + 2\gamma_1\gamma_{n-1}\cos(n-2)\theta + \ldots \qquad (3)$$
$$\ldots + \begin{cases} 2\gamma_{(n-1)/2}\gamma_{(n+1)/2}\cos\theta & \text{for odd } n \\ \gamma_{n/2}^2 & \text{for even } n \end{cases}$$

where γ_n are the coefficients calculated by the following formulas

$$\gamma_0 = 1, \quad \gamma_n = \binom{n+\nu-1}{n} = \frac{\nu(\nu+1)\ldots(n+\nu-1)}{n!}, \quad n = 1,2,..N; \quad \nu > 0$$

It is easy to see that the first-kind Chebyshev polynomials are cosine polynomials. Hence, the associated Legandre functions can be expressed through the trigonometric polynomials. In this case, the real-valued function $f(\theta, \lambda)$, determined on the surface of a sphere Ω ($0 \le \theta \le \pi$, $0 \le \lambda \le 2\pi$) and integrable with some weight (that is, $f(\theta, \lambda) \in L_2(\Omega)$), can be expanded in Fourier series using the fast Fourier transformation algorithm [3]. When this algorithm is used to calculate the Fourier coefficients over latitude and over the whole globe, the number of arithmetic operations becomes as small as $O(N*lnN)$ and $0(N^2*lnN)$, respectively.

References

1. ***Hobson E*.W. The Theory of Spherical and Ellipsoidal Harmonics. Cambridge,1931.**
2. ***Gabor Szego.* Ortogonal polynomials. Publ. Amer. Math. Soc., 1959**
3. ***Frolov A.V. and V.I. Tsvetkov. "On harmonic analyses of real-valued functions on a sphere" – Submitted to the Journal of Numerical Mathematics and Mathematical Physics, December 2003 .***